Kinetics of self-induced aggregation of Brownian particles: Non-Markovian and non-Gaussian features

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In this paper we have explored a model for self-induced aggregation of Brownian particles incorporating non-Markovian and non-Gaussian character of the associated random noise processes. The time evolution of each individual is guided by an overdamped Langevin equation of motion with a nonlocal drift arising out of the imbalance in the local distribution of the other individuals. Our simulation results show that colored noise enhances the tendency of cluster formation. Another observation is that the critical noise variance decreases at first with increase in noise correlation time followed by an increase after exhibiting a minimum. Furthermore, in the long time limit the cluster number in the aggregation process exhibits depletion with time following a power law with an exponent which increases remarkably with non-Markovian character of the noise processes.

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I. INTRODUCTION

Formation of large spatial structures and clusters by the aggregation of small species joining each other constitutes a broad area of research in pure and applied sciences [1–4]. General properties of aggregation dynamics include microphysics of clouds and precipitation [1], the principle of polymer formation [2], different types of ecological problems [4], etc., to mention a few. An important aspect of the theory of aggregation dynamics concerns the realm of biological systems in which cooperative activity among individuals usually involves social behavior [4,5]. Although its underlying mechanism, in general, remains unknown, a number of attempts have been made to establish a rigorous and quantitative basis of the emergence of cooperation (CP) among individuals [6–18]. For example, considering a one-dimensional (1D) model of clustering the effect of environment on the cooperation mechanism has been investigated in Refs. [19–21]. Here the authors have considered white noise as the random perturbation of the environment. The random fluctuations in the social communities, on the other hand, are, in general, nonthermal in origin. They may appear as a result of complicated inherent dynamics and therefore the noise of nonthermal origin may be non-Gaussian and correlated in characteristics. The correlated noise has played an important role in the context of CP which is related to aggregation dynamics since correlation (CR) among the particles increases with increase of noise correlation time (CT). Keeping this in mind we have extended the study of the aggregation dynamics in a self-induced 1D model [21] to non-Markovian and non-Gaussian regimes. Our aim is to explore how the dynamics of aggregation to form clusters depends on non-Markovian and non-Gaussian properties of the noise processes. The study is motivated by the recent experimental and theoretical observation on neural network and sensory systems [22,23] which offer strong indication that the noise sources in these systems could be non-Gaussian. The noise of biological origin in many cases is of nonlinear dynamical origin which is correlated and non-Gaussian in character, specifically, in the context of evolution [22,24]. The role of colored non-Gaussian noise in the barrier crossing dynamics, the stochastic resonance, and complex networks has also been explored by several authors [25].

II. THE MODEL

Consider a system consisting of N individuals which change their state x_i according to a majority rule. x_i denotes the reputation score of the ith member or the position in a possible chemotaxis description or some other amplitude characterizing the role of an individual within the framework of population biology. We assume that each individual changes x_i by the following stochastic equation of motion:

$$\dot{x}_i = v(x_i) + \eta_i(t). \tag{1}$$

Here v denotes the drift, which is given by the following expression:

$$v(x_i) = \lambda \frac{w_+(x_i,t) - w_-(x_i,t)}{w_+(x_i,t) + w_-(x_i,t)}, \tag{2}$$

where $w_\pm$ are defined by

$$w_\pm(x_i) = \sum_j \Theta[\pm(x_j - x_i)]\exp(-\alpha|x_j - x_i|). \tag{3}$$

Θ in the above equation is the unitary step function. Equations (1) and (2) reveal that the velocity at which an individual decides to move to the left or to the right depends on the difference $w_+ - w_-$. The magnitude of $w_\pm$ depends on the exponential weight with a coefficient $\alpha = 1/r_0$; r_0 specifies the range up to which one individual still perceives the presence and the influence of the other member of the group. So according to the above description aggregation results from preferable migration of individuals depending on the sign of v. The term $w_+ + w_-$ in the denominator of Eq. (2) presents the normalization factor and so the velocity is bounded within a range $[-\lambda, \lambda]$.

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We assume that $\eta_i(t)$ in Eq. (1) is a colored noise which may be of Gaussian or non-Gaussian type depending on the situation. η_i and η_j are uncorrelated ($\langle \eta_i(t)\eta_j(t)\rangle=0$, for $i \neq j$). The non-Gaussian noise can be generated from the solution of the following Langevin equation [26]:

$$\dot{\eta}_i = -\frac{1}{\tau}\frac{d}{d\eta}V_p(\eta_i) + \frac{\sqrt{D}}{\tau}\zeta_i(t), \tag{4}$$

where $\zeta_i(t)$ being a standard Gaussian noise of zero mean and its two-time correlation is given by

$$\langle \zeta_i(t)\zeta_i(t')\rangle = 2\delta(t-t') \tag{5}$$

and

$$V_p(\eta_i) = \frac{D}{\tau(p-1)}\ln[1+(\alpha_1(p-1)\eta_i^2/2)]. \tag{6}$$

Here the form for noise η_i allows us to control the departure from the Gaussian behavior easily by changing a single parameter p. D and τ are noise parameters related to the noise intensity and the correlation time of η_i, respectively. The parameter α_1 in Eq. (6) is defined as

$$\alpha_1 = \frac{\tau}{D}. \tag{7}$$

Now we consider two different situations. For $p=1$, Eq. (4) becomes

$$\dot{\eta}_i = -\frac{\eta_i}{\tau} + \frac{\sqrt{D}}{\tau}\zeta_i(t), \tag{8}$$

i.e., the time evolution equation of the Ornstein-Uhlenbeck noise process [28] for which the correlation function $\langle \eta_i(t)\eta_i(0)\rangle$ decays exponentially

$$\langle \eta_i(t)\eta_i(0)\rangle = \frac{D}{\tau}\exp\left(-\frac{t}{\tau}\right). \tag{9}$$

Thus τ is the correlation time of the Ornstein-Uhlenbeck noise. In the next step we consider the case where $p>1$. For this case the stationary properties of the noise η_i, including the time correlation function, have been studied in Ref. [27] and here we summarize the main results. The stationary probability distribution is given by

$$P(\eta_i) = \frac{1}{Z_p}\left[1+\alpha_1(p-1)\frac{\eta_i^2}{2}\right]^{-1/(p-1)}, \tag{10}$$

where Z_p is the normalization factor and is given by

$$Z_p = \int_{-\infty}^{\infty} d\eta_i\left[1+\alpha_1(p-1)\frac{\eta_i^2}{2}\right]^{-1/(p-1)}$$
$$= \sqrt{\frac{\pi}{\alpha_1(p-1)}}\frac{\Gamma[1/(p-1)-1/2]}{\Gamma[1/(p-1)]}. \tag{11}$$

Γ in the above equation signifies the Gamma function. This distribution can be normalized only for $p<3$. Since the above distribution function is an even function of η_i, the first moment, $\langle \eta_i\rangle$, is always equal to zero, and the second moment is given by

$$\langle \eta_i^2(p)\rangle = \frac{2D}{\tau(5-3p)}, \tag{12}$$

which is finite only for $p<5/3$. Furthermore, for $p<1$, the distribution has a cutoff and it is only defined for

$$|\eta_i| < \eta_{ic} \equiv \sqrt{\frac{2D}{\tau(1-p)}}. \tag{13}$$

Finally, the correlation time of non-Gaussian noise τ of the stationary regime of the process $\eta_i(t)$ diverges near $p=5/3$ and it can be approximated over the whole range of values of p as

$$\tau_p \simeq 2\tau/(5-3p). \tag{14}$$

Clearly, when $p\to 1$, we recover the limit of a Gaussian colored noise, the Ornstein-Uhlenbeck process, since in this limit the term in the square brackets of Eq. (10) can be written as

$$1+\alpha_1(p-1)\frac{\eta_i^2}{2} = \exp\left(\alpha_1(p-1)\frac{\eta_i^2}{2}\right) \tag{15}$$

and, therefore, Eq. (9) becomes

$$P(\eta_i) = \frac{1}{Z_1}\exp(-\alpha_1\eta_i^2/2), \tag{16}$$

with

$$Z_1 = \sqrt{\pi/\alpha_1}. \tag{17}$$

Equation (12) shows that for a given external noise strength D and noise correlation time τ the variance of the non-Gaussian noise is higher than that of the Gaussian noise for $p>1$, i.e.,

$$\langle \eta_i^2(p)\rangle > \langle \eta_i^2(1)\rangle. \tag{18}$$

Similarly Eq. (14) implies that $\tau_p>\tau$ for $p>1$.

III. RESULTS AND DISCUSSION

Based on the above-mentioned model we have investigated the aggregation dynamics numerically. To follow the dynamics of each individual present in the system N coupled stochastic equations [Eq. (1)] are solved along with the equation for noise process (4) simultaneously using the standard Heun's algorithm. We thus proceed as follows. In the first step the calculated quantities are

$$k_i = h\{v[x_i(t)] + \eta_i(t)\}, \tag{19}$$

$$l_i = hV_p[\eta_i(t)], \tag{20}$$

and

$$m_i = \sqrt{hD/\tau^2}\zeta_i(t). \tag{21}$$

h is the integration step length. In the above equation we incorporate the magnitude of $\zeta_i(t)$ using the well-known

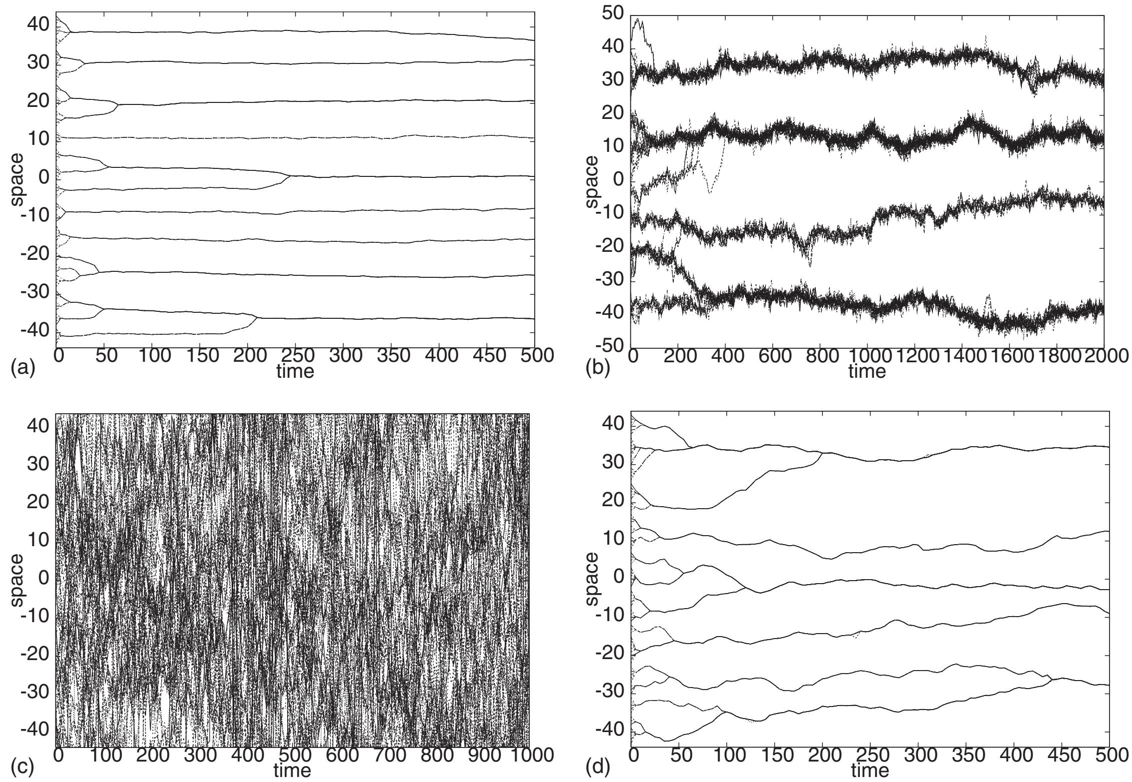

FIG. 1. This figure refers to the system dynamics with 100 particles starting from uniformly distributed initial conditions in the spatial range $[-50,+50]$. The parameter set for subfigures (a) $p=1$, $\tau=0.01$, $D=0.01$, (b) $p=1$, $\tau=0.01$, $D=0.1$, (c) $p=1$, $\tau=0.01$, $D=3.0$, (d) $p=1$, $\tau=10.0$, $D=3.0$. $\alpha=1$ and $\lambda=1$ for all the subfigures.

Box-Muller-Wiener method [30–33]. However, using the above quantities we update x_i and η_i as

$$y_i = x_i(t) + k_i \tag{22}$$

and

$$z_i = \eta_i(t) + l_i + m_i. \tag{23}$$

From the above equations the position of the ith particle and random perturbation on it (η_i) at time $t+h$ can be determined from the given values of x_i and η_i at time t as

$$x_i(t+h) = x_i(t) + \frac{h}{2}\{v[x_i(t)] + v(y_i) + \eta_i(t) + z_i\} \tag{24}$$

and

$$\eta_i(t+h) = \eta_i(t) + \frac{h}{2}\{V_p[\eta_i(t)] + V_p(z_i)\} + m_i. \tag{25}$$

Thus Heun's method is a stochastic Runge-Kutta-type method and it reduces to the second order Runge-Kutta method in the absence of noise [34]. In this numerical scheme we have considered the sum in Eq. (3) over the whole population. A very small time step (h) of 0.01 for numerical integration has been used. For the initial coordinates we have assumed that at $t=0$ all the particles are uniformly distributed in a one-dimensional box of fixed size L. The number of particles per unit length is called particle density and it is represented by ρ. The boundary condition used in the present calculation is the same as in Ref. [21]. If a particle close to the boundaries escapes the box, it is reinjected on the other side. The authors of Ref. [21] checked that it gives the same result as with the periodic boundary condition.

To have an idea about the role of noise strength, noise correlation time, and other noise parameters on the cluster formation dynamics we plot the position of all the particles vs time, t, in Fig. 1. It shows that the cluster formation tendency as well as size of the cluster increases as the noise strength is raised up to a critical value [Figs. 1(a) and 1(b)]. To check whether the cluster number (CN) converges toward a stationary value we have specially plotted in Fig. 1(b) the trajectory for a very long time. It implies that CN really converges toward a stationary limit. However, beyond a critical value of the noise strength all the individual particles exhibit the normal Brownian motion [Fig. 1(c) exhibits almost homogeneous distribution of particles over the space]. However, an increase in the noise correlation time for a fixed noise strength corresponding to the homogeneous distribution of particles as in Fig. 1(c) results in cluster formation. This is shown in Fig. 1(d). Thus colored noise can induce the cluster formation. Furthermore, a generic clustered configuration in Fig. 1(d) obtained from colored Gaussian noise becomes disordered as shown in Fig. 1(c) when one switches from Gaussian to non-Gaussian nature of noise. This is reminiscent of a phase transition from the clustered state to a homogeneous state. The cluster formation induced by colored noise also can be demonstrated by calculating the structure factor $[S(q,t)]$. It is defined as

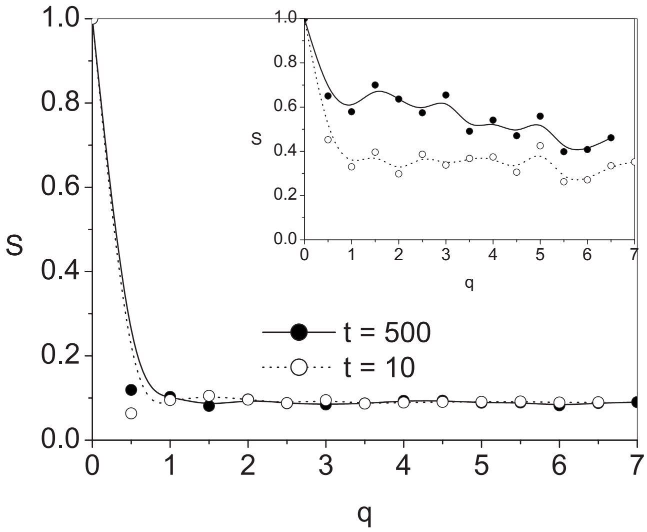

FIG. 2. Plot of the structure factor S vs wave vector $\mathbf{q}$ at different times for the parameter set $\alpha=1$, $\lambda=1$, $\rho=1$, $\tau=0.01$, $D=3.0$, and $p=1$. In the inset, same plot but $\tau=10.0$.

$$S(q,t) = \frac{1}{N}\left|\sum_{k=1}^{N} \exp[iqx_k(t)]\right|, \quad (26)$$

where q is the wave vector. Here $q \to 0$ is a trivial limit for $x_k \neq 0$. However, $S(q,t)$ measures the degree of cluster formation on determining the phase relation among the particles. If there is one cluster then all the particles are in the same phase and then the above sum reduces to $N \exp(iqx_k)$. Thus for a single cluster the structure factor is one and it is zero for the homogeneous distribution of particles around the origin. In Fig. 2 we have plotted $S(q,t)$ as a function of q at different times corresponding to the parameter sets of Figs. 1(c) and 1(d). It further confirms that noise correlation can induce the cluster formation (see the inset of Fig. 2). The structure factor is slowly decaying function of q with a quasioscillating behavior because of accumulation of more than one cluster at different positions for each time. Since before the stationary state both the locations and the number of clusters are different at different times, there is no scaling form between the structure factors calculated at different times.

We now return to rationalize the facts observed in Fig. 1. The cluster formation is a result of cooperation among the particles. The CP mechanism is effected by the dual role of noise: pushing particles toward the influence zone of a larger cluster, but also taking particles away from clusters. If the diffusion is weak then the first role dominates over the other and the noise assists to form a bigger cluster. However, if the noise strength is very high, the diffusion dominates over the cooperative effect and we observe the phase where particles are distributed almost uniformly over the space instead of cluster formation. Next we come to the point of how noise correlation time can induce cluster formation. With increase of noise correlation time the variance of noise decreases and consequently the diffusion of the particles is suppressed due to the non-Markovian nature of the noise. In addition, the noise correlation time strongly affects the drift term in the dynamics [28,29]. The drift term in the present problem accounts for the extent of cooperation among the particles which leads to cluster formation. Thus it is apparent that the colored noise-induced nucleation is a result of the extension of the cooperation and the correlation among the particles as well as suppression of the diffusive nature of the particles with increase of noise correlation time. Finally, we trace the origin of how the cluster formation is suppressed by the non-Gaussian noise. For a given noise strength the variance of the non-Gaussian noise is much higher compared to the Gaussian noise [see Eq. (18)]. As a result the diffusion may dominate over the cooperative effect for non-Gaussian noise and there is no cluster formation. On the other hand, for the same noise strength the cluster formation is possible for Gaussian noise due to weak diffusion compared to non-Gaussian noise. This points toward a phase-transition-like phenomenon as one switches from non-Gaussian to Gaussian noise or vice versa. It may be noted here that the results are independent of the initial condition except for the case of uniform distribution over the 1D box in the absence of noise, since the cooperation among the particles only depends on their distance [Eq. (3)]. Another point to be mentioned is that one individual perceives the presence and the influence of the other member of the group over a finite distance and therefore at long time almost a finite number of nucleation centers as well as clusters are observed. We may call this state a "quasistationary" (QS) state. The notation $N_c(t_s)$ is used to imply the cluster number at quasistationary state. t_s corresponds to the time after which the system reaches to the QS state. Using the quantity $N_c(t_s)$ we have defined an order-parameter-like quantity

$$\phi = \frac{N_c(t_s)}{N_c(t=0)}, \quad (27)$$

where $N_c(t=0)$ is the initial cluster number. We define a cluster as a set of particles with mutual distance $|x_i - x_j|$ below a given cutoff ϵ which one may call resolution. However, at low noise strength, $\phi \to 0$ and it becomes close to unity when noise strength is very high. To check whether the transition (from the clustered state to the homogeneous state) is continuous or not we have plotted ϕ vs noise strength for different box lengths in Fig. 3 for $t_s=1000$. It shows that for a larger box the transition is smooth and continuous. Noise-assisted clustering is manifested in Fig. 3 through the initial decrease of ϕ with increase in noise strength for large box length. The value of the order parameter in Fig. 3 at low noise strength is close to zero. It implies that the chosen quasistationary time for Fig. 3 is reasonably good with respect to the very slow rate of decrease of the cluster number at long time. To check it we have calculated ϕ for box length $L=1000$ at $t_s=1200$ and compared its value with that of $t_s=1000$. This analysis is presented in the inset of Fig. 3. It shows that there is a very little decrease in ϕ, particularly at low noise strength and the critical value of the noise strength around which the continuous transition takes place remains unchanged. At the same time, the inset implies that the robustness of Fig. 3 is satisfactory with respect to the change of the t_s and this figure would remain almost the same over a

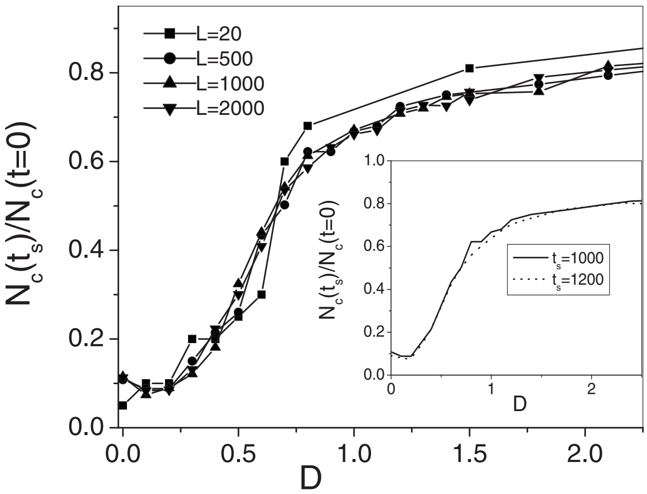

FIG. 3. Plot of the ratio of number of clusters at quasistationary state to the initial time vs noise strength for Gaussian noise for different one-dimensional box lengths for the parameter set $\alpha=1$, $\lambda=1$, $\rho=1$, $\epsilon=0.1$, $\tau=1$, $t_s=1000$, and $p=1$. The inset figure presents the same plot for $L=1000$ and $t_s=1200$.

long time. However, from Fig. 3 one may determine the critical noise variance ($\langle \eta_p^2 \rangle_c = \frac{2D_c}{\tau(5-3p)}$) at which the continuous transition occurs. D_c is the value of threshold noise strength around which the sharp change of the order parameter ϕ takes place and it is estimated numerically from the plot of the order parameter vs D for a given noise correlation time τ. To determine the value of D_c we have chosen the quasistationary time such that it is almost invariant on further increase of t_s. We have checked that $t_s=1000$ is a good limit for Fig. 4, where we have presented how $\langle \eta_p^2 \rangle_c$ depends on the noise correlation time. The initial fall of the critical variance is due to slow increase of D_c with τ. As τ becomes large the correlation as well as the cooperation among the particles grows and that leads to an increase in D_c at a faster rate. As a result the critical variance first decreases followed by an increase exhibiting a minimum. The appearance of a minimum at low τ for non-Gaussian noise is a consequence of Eq. (14). Because of higher effective noise correlation time for the non-Gaussian noise compared to the Gaussian noise,

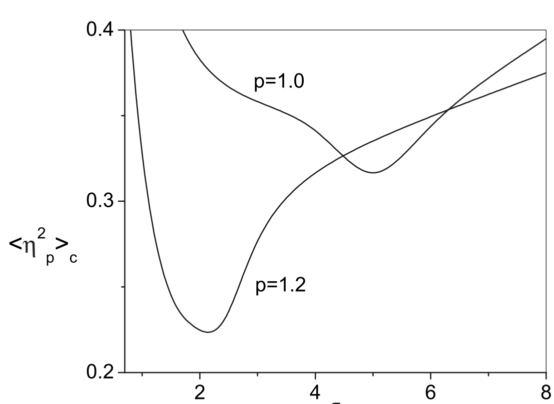

FIG. 4. Plot of critical noise variance vs noise correlation time for the parameter set $\alpha=1$, $\lambda=1$, $\rho=1$, $\epsilon=0.1$, $t_s=1000$, and $L=1000$.

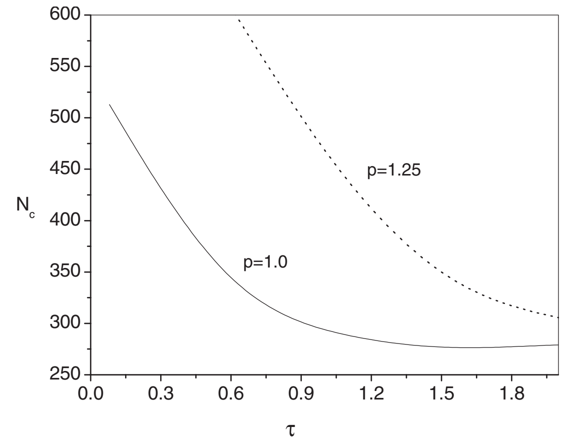

FIG. 5. Plot of the variation of the cluster number as a function of the correlation time of the noise for different values of the non-Gaussian parameter (p) for the parameter set $\lambda=1$, $\alpha=1$, $\rho=4$, $\epsilon=0.25$, $t_s=1000$, and $D=0.3$.

CR and CP become important in the dynamics at low τ for the former (compared to the corresponding latter case).

In the next step we have investigated the variation of cluster number $N_c(t_s)$ at the quasistationary state with noise correlation time (τ) and the results are plotted in Fig. 5. It exhibits the cluster number rapidly falling with τ for the Gaussian noise when compared to the corresponding case for the non-Gaussian noise. As the correlation and the cooperation among the particles increases with increasing τ the bigger cluster is formed for larger noise correlation time. Because of that the cluster number reduces for both Gaussian and non-Gaussian noises as τ grows. The slow decrease of cluster number for non-Gaussian noise (compared to the case for Gaussian noise) is due to higher noise variance (after a certain value of noise variance it is difficult to form a cluster).

The above discussions raise an important question: how does the aggregation kinetics depend on the noise correlation time? To this end we have explored the aggregation processes through a set of simulation studying on a system involving 10^4 particles. We choose the initial conditions corresponding to the clustered phase in the phase diagram presented in Fig. 4. Our observation is that (Fig. 6) both in the Markovian and the non-Markovian limits the cluster number rapidly decreases with time at the short time regime and beyond that it slowly varies. For a detailed analysis of the influence of the noise correlation and non-Gaussian parameter on the rate of aggregation processes we set an approximate power law of the following form:

$$N_c(t) \sim t^{-z}. \tag{28}$$

This type of algebraic decay law is valid for all τ and p values provided t is long enough (see the inset of Fig. 6). From Fig. 6 it is surprising to note that the exponent z in the power law increases more than fifty percent compared to the Markovian case for $\tau=1.0$. We are thus led to believe that the noise correlation time enhances the cooperative effect in the aggregation dynamics through modification of the drift term as well as by reducing the noise variance for the large noise

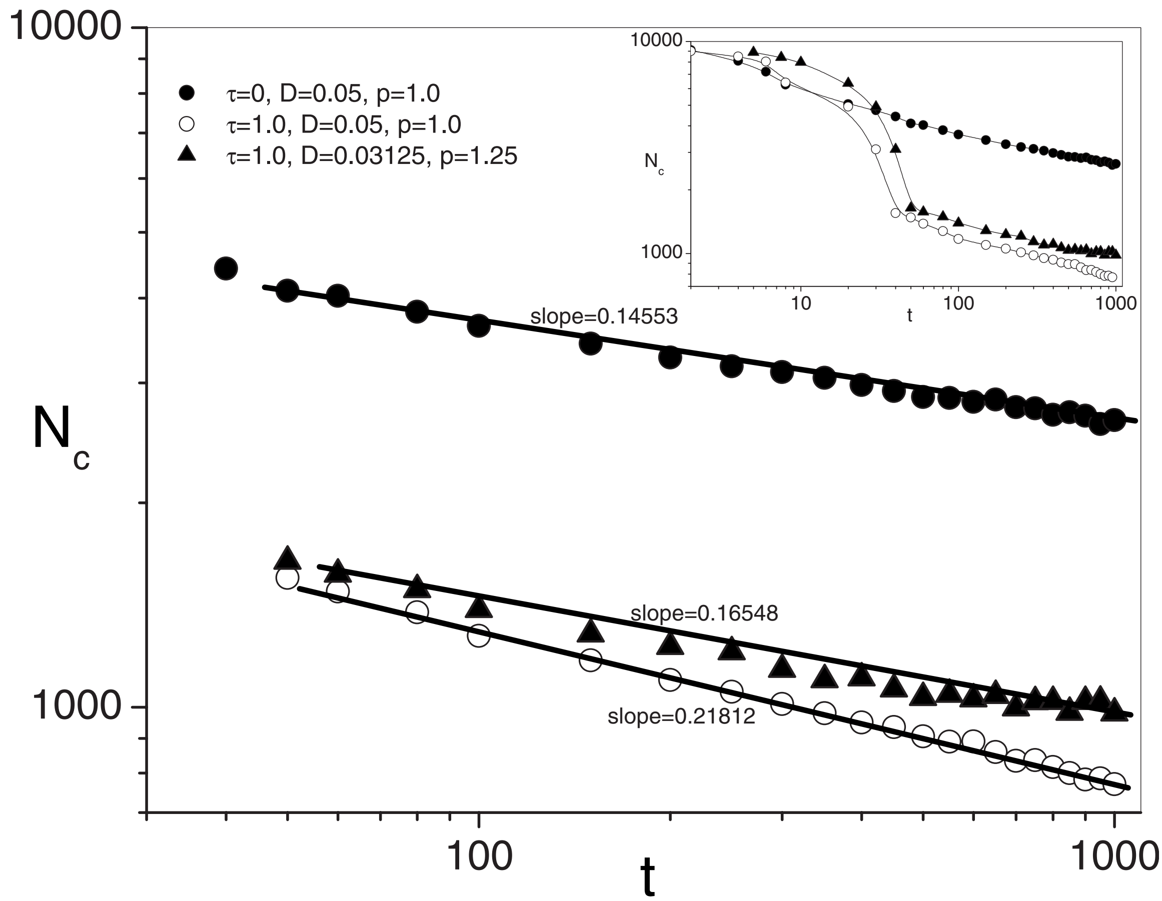

FIG. 6. Plot of the variation of the cluster number as a function of time at long time limit for different values of the noise correlation time (τ) and the non-Gaussian parameter (p) for the parameter set $\lambda=1$, $\alpha=1$, $\rho=1$, and $\epsilon=0.25$. In the inset same plot which covers both the short time as well as long time limits.

strength. We mention, in passing, that the exponent z is a little bit smaller for the colored non-Gaussian noise compared to the Gaussian case as a result of higher noise variance for the former for a given noise strength.

IV. CONCLUSION

Based on the numerical simulation of stochastic dynamics associated with colored non-Gaussian noise we have investigated a model for self-induced aggregation kinetics. Such a type of self-induced model is used to deal with a certain class of problems in social biological problems. In this model, each individual is represented by a Brownian particle, where drift velocity depends on the population imbalance perceived by a single individual. Based on a set of numerical simulation studies we have examined the effect of the noise correlation and non-Gaussian character of the noise in the self-induced aggregation dynamics. Our main observation includes the following points.

(i) The aggregation of particles is facilitated by increase of noise correlation time (Figs. 1 and 2). The aggregation may disappear if one switches from Gaussian to non-Gaussian noise for a given noise strength and correlation time.

(ii) The change of state from homogeneous distribution of particles to the clustered state is continuous.

(iii) The critical noise variance first decreases with an increase of noise correlation time followed by an increase and a minimum. The minimum appears at low noise correlation time for non-Gaussian noise compared to the case for Gaussian noise.

(iv) The cluster number at quasisteady state decreases with increase of noise correlation time.

(v) The cluster formation kinetics follows a power law for the variation of cluster number with time given by $N_c(t) \sim t^{-z}$ at long time. The exponent z remarkably increases for the non-Markovian case.

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